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INFINITE DIMENSIONAL DIFFUSION PROCESSES WITH SINGULAR INTERACTION

BY

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ABSTRACT. – An infinite system of hard spheres in $\mathbb{R}^d$ undergoing Brownian motions and submitted to a smooth pair potential is studied. It is modelized by an infinite-dimensional Stochastic Differential Equation with a local time term. We prove existence and uniqueness of such a diffusion process, and also that Gibbs states are reversible measures. © 2000 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous étudions un système infini de sphères dures dans $\mathbb{R}^d$ diffusant de façon brownienne et soumises de plus à un potentiel régulier d'interaction par paires. Nous montrons existence et unicité des solutions de l'équation différentielle stochastique infini-dimensionnelle (contenant un terme de temps local) modélisant ce systéme et prouvons que les mesures de Gibbs sont réversibles. © 2000 Éditions scientifiques et médicales Elsevier SAS

Mots Clés: Equation Différentielle Stochastique, Potentiel de sphères dures, Mesure de Gibbs, Mesure réversible

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1. Introduction

In [2], Kolmogorov discovered an important connection between time reversible diffusions and Gibbs measures in the context of finite-dimensional processes. Here, we extend this problematic to the case of some infinitely many interacting diffusions. More precisely, we consider an infinite system of hard spheres in $\mathbb{R}^d$ undergoing Brownian motions and submitted to the influence of a smooth finite range pair potential.

On one side, systems of infinite Brownian particles (i.e., spheres with radius 0) with smooth pair interaction have been treated in a pioneer work by R. Lang [3], and then by J. Fritz [1]. On another side, a system of infinitely many Brownian spheres without external potential was studied by H. Tanemura [8]. In the present paper, the model is a mixture of both models. We deal with Brownian motions submitted to the sum of a hard core potential and a smooth finite range pair potential. We prove existence and uniqueness of a solution of corresponding equation $(\mathcal{E})$ stated in Section 3, under the condition that the initial density of spheres is small enough (condition $z < z_c$ in Theorem 2.3, which is natural from a physical point of view). We also obtain that Gibbs initial measures are reversible.

Let us just mention that the law of such processes could be studied from Dirichlet form point of view. H. Tanemura [9] did it for pure hard core systems. See also the work of H. Osada [4] and M. Yoshida [10] for a large class of potentials.

After a second section where notations are introduced, in Section 3 we present the infinite dimensional equation $(\mathcal{E})$ and we state the results. The sequence of approximating solutions is built in Section 4. In Section 5 we prove technical estimates needed in Section 6, for the convergence of the approximations. Finally, Section 7 is devoted to complete the proof of the main results.

2. Configuration spaces and notations

The particles we deal with in the present paper evolve in $\mathbb{R}^d$, for a fixed $d \geqslant 1$, endowed with the euclidian norm denoted by $|\ |$. $B(x_0, \rho)$ will denote the closed ball centered in $x_0 \in \mathbb{R}^d$ with radius ρ and more generally, for any subset A in $\mathbb{R}^d$, we let

$$B(A,\rho)=\left\{x\in\mathbb{R}^d \text{ such that } d(x,A)\leqslant\rho\right\},$$

where $d(x,A)$ denotes the (euclidian) distance between A and x. By simplicity, the volume of a subset A in $\mathbb{R}^d$ is also denoted by $|A|$.

The modelization of point configurations may be done in two equivalent ways: the first possibility is to represent an n points configuration in $\mathbb{R}^d$ as a subset (with multiplicity) of cardinal n in $\mathbb{R}^d$, that is as an equivalence class on $(\mathbb{R}^d)^n$ under the action of the permutation group S_n on $\{1,\ldots,n\}$. The second possibility is to modelize it as a point measure $\sum_{i=1}^n \delta_{\xi_i}$.

More generally, the set of all point configurations in $\mathbb{R}^d$ will be the set $\mathcal{M}$ of all point Radon measures on $\mathbb{R}^d$, that is

$$\mathcal{M}=\left\{\xi=\sum_{i\in I\subset\mathbb{N}}\delta_{\xi_i} \text{ such that } \xi_i\in\mathbb{R}^d \text{ and } \forall\Lambda \text{ compact in } \mathbb{R}^d,\ \xi(\Lambda)<+\infty\right\},$$

endowed with the topology of vague convergence.

We introduce the following notations.

- For $A\subset\mathbb{R}^d$, N_A is the counting variable on $\mathcal{M}$:

$$N_A(\xi)=\mathrm{Card}\{i\in\mathbb{N},\ \xi_i\in A\}.$$

- For $A\subset\mathbb{R}^d$, $\mathcal{B}_A$ is the σ-algebra on $\mathcal{M}$ generated by the sets $\{N_B=n\}$, $n\in\mathbb{N}$, $B\subset A$, B bounded.
- π (respectively π_A) is the Poisson process on $\mathbb{R}^d$ (respectively on A) with intensity measure the Lebesgue measure dx (respectively $dx|_A$).

The particles we deal with in this paper are not reduced to points but are hard spheres of radius $r/2$, for a fixed $r>0$. So the set of "allowed configurations" is the following subset of $\mathcal{M}$:

$$\mathcal{A}=\left\{\xi\in\mathcal{M} \text{ such that } \forall i\neq j\ |\xi_i-\xi_j|\geqslant r\right\}.$$

Let us denote $\mathcal{C}(\mathbb{R}^+,\mathcal{M})$ the set of continuous $\mathcal{M}$-valued paths on $\mathbb{R}^+$, endowed with the topology of uniform convergence on each compact time interval. The subset of all "allowed" configuration evolutions is

$$\mathcal{C}\big(\mathbb{R}^+, \mathcal{A}\big) = \big\{X \in \mathcal{C}(\mathbb{R}^+, \mathcal{M}) \text{ such that } \forall t \geqslant 0 \ X(t) \in \mathcal{A}\big\}.$$

Remark 2.1. – We study here the evolution of a particles configuration under the influence of an interaction potential with finite range R. Then a fixed particle can interact with at most a finite number $\overline{N}$ of particles, which is clearly bounded by $(R + r/2)^d/(r/2)^d$.

3. Statement of the results

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that each $\mathcal{F}_t$ contains all P-negligible sets and let $(W_i(t), t \geqslant 0)_{i\in\mathbb{N}}$ be a family of $\mathcal{F}_t$-adapted independent d-dimensional Brownian motions.

We consider the following infinite system of stochastic equations:

$$(\mathcal{E})\begin{cases} \text{For } i \in \mathbb{N}, t \in \mathbb{R}^+, \\ X_i(t) = X_i(0) + W_i(t) - \dfrac{1}{2}\displaystyle\sum_{j\in\mathbb{N}} \int_0^t \nabla\varphi\big(X_i(s) - X_j(s)\big)\, ds \\ \qquad\qquad + \displaystyle\sum_{j\in\mathbb{N}} \int_0^t \big(X_i(s) - X_j(s)\big)\, dL_{ij}(s), \end{cases}$$

where

- $(X_i(t),\ t \geqslant 0)_{i\in\mathbb{N}}$ is a continuous $\mathcal{A}$-valued process, i.e. satisfying

$$\big|X_i(t) - X_j(t)\big| \geqslant r \quad \text{for } t \geqslant 0 \text{ and } i \neq j;$$

- φ is a smooth stable pair potential with finite range R;
- $(L_{ij}(t),\ t \geqslant 0)_{i,j\in\mathbb{N}}$ is a family of non-decreasing $\mathbb{R}^+$-valued continuous processes satisfying:

$$L_{ij}(0) = 0, \quad L_{ij} \equiv L_{ji} \quad \text{and} \quad L_{ij}(t) = \int_0^t \mathbb{1}_{|X_i(s)-X_j(s)|=r}\, dL_{ij}(s).$$

By convention, we will always take $L_{ii} \equiv 0$.

A solution of the system $(\mathcal{E})$ is a family $(X_i(t), L_{ij}(t),\ t \geqslant 0,\ i, j \in \mathbb{N})$ (or simply $(X_i(t),\ t \geqslant 0)_{i\in\mathbb{N}})$ of processes such that the equation $(\mathcal{E})$ and the above conditions are satisfied.

Then the interaction between the coordinates of the process derives from the action of two potentials:

- φ a pair potential, function on $\mathbb{R}^d$ of class $\mathcal{C}^2$ with finite range $R > r$, i.e., satisfying $\varphi(x) = 0$ if $|x| \geqslant R$ and $\varphi(x) = \varphi(-x)$ (then $\nabla\varphi(0) = 0$). We denote by $\underline{\varphi}$ the following lower bound:

$$\underline{\varphi} = \inf_{|x|\geqslant r} \varphi(x) \leqslant 0.$$

- ψ a r-diameter hard core pair potential, i.e., such that $\psi(x) = +\infty$ if $|x| < r$ and $\psi(x) = 0$ otherwise.

We now define the set $\mathcal{G}(z)$ of Gibbs states associated to the potential $\varphi + \psi$ with activity parameter $z \in \mathbb{R}^+$. For each compact subset Λ of $\mathbb{R}^d$, let us define a local density function by:

$$(1)\quad f_\Lambda(\xi|\eta) = \frac{z^{N_\Lambda(\xi)}}{\mathrm{e}^{-|\Lambda|} Z^{\Lambda,\eta}} \mathbb{1}_{\mathcal{A}}(\xi_\Lambda \eta_{\Lambda^c}) \times \exp\Bigg(-\frac{1}{2} \sum_{\substack{\xi_i,\xi_j\in\Lambda \\ i\neq j}} \varphi(\xi_i - \xi_j) - \sum_{\substack{\xi_i\in\Lambda \\ \eta_j\in\Lambda^c}} \varphi(\xi_i - \eta_j)\Bigg),$$

where $Z^{\Lambda,\eta}$ is a renormalizing constant determined by

$$Z^{\Lambda,\eta} = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} \mathbb{1}_{\mathcal{A}}(\xi_\Lambda \eta_{\Lambda^c}) \times \exp\Bigg(-\sum_{1\leqslant i<j\leqslant n} \varphi(\xi_i - \xi_j) - \sum_{\substack{1\leqslant i\leqslant n \\ \eta_j\in\Lambda^c}} \varphi(\xi_i - \eta_j)\Bigg) d\xi_1 \cdots d\xi_n,$$

and $\xi_\Lambda \eta_{\Lambda^c}$ is the element of $(\mathbb{R}^d)^{\mathbb{N}}$ which coincides with ξ on Λ and with η on Λ^c.

DEFINITION 3.1. – *A probability measure μ on $\mathcal{M}$ belongs to the set $\mathcal{G}(z)$ of Gibbs states with activity z and associated potential $\varphi + \psi$ if and only if, for each compact subset $\Lambda \subset \mathbb{R}^d$,*

$$d\mu(\xi|\mathcal{B}_{\Lambda^c})(\eta) = f_\Lambda(\xi|\eta)\, d\pi_\Lambda(\xi) \quad \textit{for } \mu\textit{-a.e. } \eta.$$

The set $\mathcal{G}(z)$ is convex and compact. Since $\varphi + \psi$ is a potential which is superstable and lower regular in the sense of Ruelle [5], $\mathcal{G}(z)$ is non empty but the question if $\mathcal{G}(z)$ is reduced to only one measure for z large enough is still an open problem.

The main results of this paper are the following theorems, proved in Section 6.

THEOREM 3.2. – *The continuous gradient system* $(\mathcal{E})$ *admits a solution with values in* $\mathcal{A}$ *for all initial configuration which belongs a.s to a set* $\underline{\mathcal{A}} \subset \mathcal{A}$ *defined by Eq.* (20). *This solution, denoted by* $(X^\infty(t), t \geqslant 0)$, *is Markovian and is unique as element of* $\mathcal{C} \subset \mathcal{C}(\mathbb{R}^+, \mathcal{A})$, *a subset of regular paths defined in Section* 7.

THEOREM 3.3. – *Let* $z < z_c$ *be a fixed activity, with*

$$z_c = \frac{\exp(2\overline{N}\underline{\varphi})}{(R^d - r^d)|B(0,1)|}.$$

The $\mathcal{A}$*-valued process solution of* $(\mathcal{E})$ *with an initial configuration distributed like* $\mu \in \mathcal{G}(z)$ *is reversible.*

Remark 3.4. – The existence of a critical value for the activity z is natural, it is related to the still open problem of percolation for the hard core continuous system. The value of z_c given here appears for technical reasons in Corollary 5.5, where a percolation type estimate is computed.

4. Construction of approximating processes

The solution of $(\mathcal{E})$ will be constructed as a limit of approximating processes $(X^l)_{l\in\mathbb{N}^*}$, by penalization. In this whole section, $l \in \mathbb{N}^*$ and $\Lambda = [-l, l]^d$ are fixed.

For an allowed configuration η concentrated outside Λ, we fix a $\mathbb{R}^+$-valued function $\psi^{l,\eta}$ on $\mathbb{R}^d$ which is $\mathcal{C}^2$ with bounded derivatives and vanishes on every $x \in \Lambda$ such that $x\eta$ is an allowed configuration, that is

$$\psi^{l,\eta}(x) = 0 \Leftrightarrow x \in \Lambda \text{ and } x\eta \in \mathcal{A} \Leftrightarrow x \in \Lambda \text{ and } d(x,\eta) \geqslant r.$$

We extend the definition of $\psi^{l,\eta}$ to configurations $\eta \in \mathcal{A}$ not necessarily belonging to $(\Lambda^c)^{\mathbb{N}}$ by $\psi^{l,\eta} = \psi^{l,\eta_{\Lambda^c}}$.

We also suppose that, for every $\eta \in \mathcal{A}$,

$$\sum_{l\in\mathbb{N}^*}\int_{\mathbb{R}^d} \mathbb{1}_{\psi^{l,\eta}(x)>0} \exp\bigl(-\psi^{l,\eta}(x)\bigr)\,dx \leqslant 1. \tag{2}$$

(Choose for example $\psi^{l,\eta}(x)=\frac{1}{5}l^3\delta(x)$ where δ is a $\mathcal{C}^2$ function with bounded derivatives which is equivalent to $d(\cdot,\Lambda-B(\eta_{\Lambda^c},r))$ on $\mathbb{R}^d$.)

For $\eta\in\mathcal{A}$ still fixed, and for $n\in\mathbb{N}^*$, let us now study the n-dimensional stochastic differential equation:

$$(\mathcal{E}_n^{l,\eta})\begin{cases} \forall i\in\{1,\dots,n\},\ \forall t\geqslant 0,\\ dX_i(t)=dW_i(t)-\dfrac{1}{2}\Bigl(\nabla\psi^{l,\eta}\bigl(X_i(t)\bigr)\\ \qquad\qquad+\displaystyle\sum_{j=1,\dots,n}\nabla\varphi\bigl(X_i(t)-X_j(t)\bigr)\\ \qquad\qquad+\displaystyle\sum_{j,\eta_j\in\Lambda^c}\nabla\varphi\bigl(X_i(t)-\eta_j\bigr)\Bigr)\,dt\\ \qquad\qquad+\displaystyle\sum_{j=1,\dots,n}\bigl(X_i(t)-X_j(t)\bigr)\,dL_{ij}(t).\end{cases}$$

It is a stochastic equation reflected in $\mathcal{A}\cap(\mathbb{R}^d)^n$ with drift $-\frac{1}{2}\nabla\beta_n^{l,\eta}$ where

$$\begin{aligned}(3)\quad &\beta_n^{l,\eta}(x_1,\dots,x_n)\\ &=\sum_{i=1,\dots,n}\Bigl(\psi^{l,\eta}(x_i)+\frac{1}{2}\sum_{\substack{j=1,\dots,n\\ j\neq i}}\varphi(x_i-x_j)+\sum_{j,\eta_j\in\Lambda^c}\varphi(x_i-\eta_j)\Bigr).\end{aligned}$$

It has a unique strong solution for each initial configuration $x\in\mathcal{A}\cap(\mathbb{R}^d)^n$ (see Theorem 5.1 of [6]). We will denote this solution by $X^{l,\eta,n}(x,\cdot)$.

PROPOSITION 4.1. – *The law, denoted by $Q_n^{l,\eta}$, of the solution of $(\mathcal{E}_n^{l,\eta})$ with initial condition $\nu_n^{l,\eta}$ is reversible, where $\nu_n^{l,\eta}$ is the finite measure defined on $(\mathbb{R}^d)^n$ by*

$$d\nu_n^{l,\eta}(x_1,\dots,x_n)=\exp\bigl(-\beta_n^{l,\eta}(x_1,\dots,x_n)\bigr)\mathbb{1}_{\mathcal{A}}(x_1,\dots,x_n)\,dx_1\cdots dx_n.$$

Proof of Proposition 4.1. – In this proof, l, η and n are fixed, hence we drop the indices and simply write β, ν, Q for $\beta_n^{l,\eta}$, $\nu_n^{l,\eta}$, $Q_n^{l,\eta}$, etc. Thanks to Theorem 5.1 of [6] again, the stochastic differential equation

$$\forall i \in \{1,\dots,n\},\ \forall t \geqslant 0,\ x \in \mathcal{A} \cap \left(\mathbb{R}^d\right)^n,$$
$$X_i(t) = x_i + W_i(t) + \int_0^t \sum_{j=1,\dots,n} \left(X_i(s) - X_j(s)\right) dL_{ij}(s),$$
$$L_{ij}(t) = \int_0^t \mathbb{1}_{|X_i(s)-X_j(s)|=r}\, dL_{ij}(s),$$

has a unique strong solution, whose law on $\mathcal{C}([0,T], \mathcal{A} \cap (\mathbb{R}^d)^n)$ is denoted by $\overline{P}^x$. It is known (see, e.g., Theorem 1 of [7]) that the measure

$$\overline{P} = \int_{\mathcal{A}\cap(\mathbb{R}^d)^n} \overline{P}^x\, dx$$

is time reversible.

If Q^x is the distribution of the unique strong solution of $(\mathcal{E}_n^{l,\eta})$ starting from x, and

$$Q = \int_{\mathcal{A}\cap(\mathbb{R}^d)^n} Q^x\, d\nu(x)$$

is the law of the solution with initial distribution ν, applying Girsanov theorem, we can compute the density:

$$\begin{aligned}\frac{dQ}{d\overline{P}}(X) = \exp\Bigg(&-\frac{1}{2}\left(\beta\left(X(0)\right) + \beta\left(X(T)\right)\right)\\ &+ \frac{1}{2}\int_0^T \sum_{i,j=1,\dots,n} \nabla_i \beta\left(X(s)\right)\left(X_i(s) - X_j(s)\right) dL_{ij}(s)\\ &+ \int_0^T \left(\frac{1}{4}\Delta\beta\left(X(s)\right) - \frac{1}{8}\left|\nabla\beta\left(X(s)\right)\right|^2\right) ds\Bigg).\end{aligned}$$

Since $\overline{P}$ and $dQ/d\overline{P}$ are invariant with respect to the time reversal, Q is time reversal invariant too, which exactly means that the solution of $(\mathcal{E}_n^{l,\eta})$ starting from ν is reversible. $\square$

The finite measure $\nu_n^{l,\eta}$ on $(\mathbb{R}^d)^n$ is an approximation of the distribution of n particles under $(\varphi+\psi)$-interaction in Λ. We now define a probability measure $\mu^{l,\eta}$ on $\bigcup_{n=0}^{+\infty}(\mathbb{R}^d)^n$ which will represent the distribution of a random number of particles in Λ, this number following a Poisson distribution with intensity measure $z\,dx$, by:

$$\text{(4)} \qquad \forall A_0 \times A_1 \times \cdots \times A_n \times \cdots \in \prod_{n=0}^{+\infty} \mathcal{B}or\big((\mathbb{R}^d)^n\big),$$
$$\mu^{l,\eta}(A_0 \times A_1 \times \cdots \times A_n \times \cdots) = \frac{1}{Z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \nu_n^{l,\eta}(A_n),$$

where $Z^{l,\eta}$ is the renormalizing constant (with the convention $\nu_0^{l,\eta}(\{\emptyset\}) = 1$).

Similarly, consider the probability measure on $\bigcup_{n=0}^{+\infty}\mathcal{C}(\mathbb{R}^+,(\mathbb{R}^d)^n)$ defined by

$$Q^{l,\eta} = \frac{1}{Z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} Q_n^{l,\eta}.$$

This reversible law is obviously the distribution of the unique strong solution $X^{l,\eta}$ of the following finite (but random)-dimensional stochastic equation:

$$(\mathcal{E}^{l,\eta}) \quad \begin{cases} \forall i \in \{1,\dots,N\} \text{ where } N := \mathrm{Card}\big(X^{l,\eta}(0)\big),\ \forall t \geqslant 0, \\ dX_i^{l,\eta}(t) = dW_i(t) - \dfrac{1}{2}\Big(\nabla\psi^{l,\eta}\big(X_i^{l,\eta}\big) + \displaystyle\sum_{j=1}^{N} \nabla\varphi\big(X_i^{l,\eta} - X_j^{l,\eta}\big) \\ \qquad\qquad + \displaystyle\sum_{j,\eta_j\in\Lambda^c} \nabla\varphi\big(X_i^{l,\eta} - \eta_j\big)\Big)(t)\,dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{N} \big(X_i^{l,\eta}(t) - X_j^{l,\eta}(t)\big)\,dL_{ij}^{l,\eta}(t), \\ \forall i,j \in \{1,\dots,N\} L_{ij}^{l,\eta} = L_{ji}^{l,\eta}, \quad L_{ij}^{l,\eta}(0) = 0, \\ L_{ij}^{l,\eta} \text{ nondecreasing and } \forall t \geqslant 0, \\ L_{ij}^{l,\eta}(t) = \displaystyle\int_0^t \mathbb{1}_{|X_i^{l,\eta}(s) - X_j^{l,\eta}(s)| = r}\, dL_{ij}^{l,\eta}(s), \quad X^{l,\eta}(0) \overset{(d)}{=} \mu^{l,\eta}. \end{cases}$$

Finally, let us randomize the external configuration η and consider the following infinite dimensional stochastic equation

$$
(\mathcal{E}^l)\quad \begin{cases}
\forall i \in \mathbb{N} \text{ such that } X_i^l(0) \in \Lambda,\ \forall t \geqslant 0,\\
dX_i^l(t) = dW_i(t) - \dfrac{1}{2}\Big(\nabla \psi^{l, X^l(0)}\big(X_i^l(t)\big) \\
\qquad\qquad + \displaystyle\sum_j \nabla\varphi\big(X_i^l(t) - X_j^l(t)\big)\Big)\,dt \\
\qquad\qquad + \displaystyle\sum_{j, X_j^l(0)\in\Lambda} \big(X_i^l(t) - X_j^l(t)\big)\,dL_{ij}^l(t),\\
\forall i \in \mathbb{N} \text{ such that } X_i^l(0) \notin \Lambda,\ X_i^l(\cdot) \equiv X_i^l(0),\\
\forall i, j,\ L_{ij}^l = L_{ji}^l,\ L_{ij}^l(0) = 0,\ L_{ij}^l \text{ nondecreasing and } \forall t \geqslant 0,\\
L_{ij}^l(t) = \mathbb{1}_{X_i^l(0)\in\Lambda}\mathbb{1}_{X_j^l(0)\in\Lambda}\displaystyle\int_0^t \mathbb{1}_{|X_i^l(s) - X_j^l(s)| = r}\,dL_{ij}^l(s).
\end{cases}
$$

For each deterministic initial configuration $X^l(0) \in \mathcal{A}$, the equation $(\mathcal{E}^l)$ has a unique strong solution (X^l, L^l) since it reduces to the dynamics of $(\mathcal{E}_n^{l,\eta})$ with $\eta = X^l(0) \cap \Lambda^c$ and $n = \mathrm{Card}(X^l(0) \cap \Lambda)$.

5. Estimates on the path set

In this section, we want to prove that the probability of trajectories of particles which interact too much, vanishes asymptotically when $l \to +\infty$. We will use this result to construct the limit of $(X^l)_l$ in the next section.

5.1. Probability of fast motion

We obtain (in Proposition 5.2) an estimate of the probability, under $Q^{l,\eta}$, that a particle moves "too fast". We first compute the probability of fast motion between two fixed bounded domains in $\mathbb{R}^d$.

Let A_0 and A_T be bounded subsets of $\mathbb{R}^d$, $\varepsilon > 0$ and $\delta \in]0, T]$.

$$
\mathcal{F}m(A_0, A_T, \varepsilon, \delta) = \big\{X \in \mathcal{C}([0,T], \mathcal{A}), \exists i \text{ such that } X_i(0) \in A_0,\\
X_i(T) \in A_T \text{ and } w(X_i, \delta, T) > \varepsilon\big\},
$$

where

$$w(X_i,\delta,T)=\sup_{\substack{|t-s|<\delta\\0\leqslant s,t\leqslant T}}\left|X_i(t)-X_i(s)\right|$$

is the usual modulus of continuity of the path X_i.

LEMMA 5.1. – *There exists a constant $C_0\geqslant 0$ depending only on T, z, R, r, d and φ, such that for each A_0, A_T bounded subsets of $\mathbb{R}^d$ and each $\varepsilon>0$, $\delta\in]0,T]$, we have*:

$$\forall l\in\mathbb{N}^*,\ \forall\eta\in\mathcal{A},$$
$$Q^{l,\eta}\big(\mathcal{F}m(A_0,A_T,\varepsilon,\delta)\big)\leqslant C_0\big(|A_0|+|A_T|\big)\frac{1}{\delta}\exp\left(-\frac{\varepsilon^2}{20\delta}\right).$$

For every $K\in\mathbb{N}^*$, $\varepsilon>0$ and $\delta\in]0,T]$, let $\mathcal{F}m(K,\varepsilon,\delta)$ be the following event:

$$\mathcal{F}m(K,\varepsilon,\delta)=\big\{X\in\mathcal{C}([0,T],\mathcal{A})\text{ such that }\exists i,\ X_i(0)\in B(0,K)\\ \text{and } w(X_i,\delta,T)>\varepsilon\big\}.$$

PROPOSITION 5.2. – *There exists a constant $C_1\geqslant 0$ depending only on T, z, R, r, d and φ, and a constant $C_2>0$ depending only on T and d, such that*:

$$\forall K\in\mathbb{N}^*,\ \forall\varepsilon>0,\ \forall\delta\in]0,T],\ \forall l\in\mathbb{N}^*,\ \forall\eta\in\mathcal{A},$$
$$Q^{l,\eta}\big(\mathcal{F}m(K,\varepsilon,\delta)\big)\leqslant C_1K^d\frac{1}{\delta}\exp\left(-C_2\frac{\varepsilon^2}{\delta}\right).$$

Proof of Lemma 5.1. – We first need an estimate of $Q_n^{l,\eta}(\mathcal{F}m(A_0,A_T,\varepsilon,\delta)\cap(\mathbb{R}^d)^n)$.

By construction, the processes:

$$\Bigg(W_i(t)=X_i^{l,\eta,n}(t)-X_i^{l,\eta,n}(0)+\frac{1}{2}\int_0^t\nabla_i\beta_n^{l,\eta}\big(X^{l,\eta,n}(s)\big)\,ds\\ -\int_0^t\sum_{j=1,\dots,n}\big(X_i^{l,\eta,n}(s)-X_j^{l,\eta,n}(s)\big)\,dL_{ij}^{l,\eta,n}(s)\Bigg)_{1\leqslant i\leqslant n}$$

and

$$\left(\widehat{W}_i(t) = X_i^{l,\eta,n}(T-t) - X_i^{l,\eta,n}(T) + \frac{1}{2}\int_{T-t}^{T} \nabla_i \beta_n^{l,\eta}\big(X^{l,\eta,n}(s)\big)\,ds \right.$$
$$\left. - \int_{T-t}^{T} \sum_{j=1,\dots,n} \big(X_i^{l,\eta,n}(s) - X_j^{l,\eta,n}(s)\big)\,dL_{ij}^{l,\eta,n}(s)\right)_{1\leqslant i\leqslant n},$$

are both n-dimensional Brownian motions on $[0,T]$ starting from 0. Remarking that

$$\forall t \in [0,T], \quad X^{l,\eta,n}(t) - X^{l,\eta,n}(0) = \frac{1}{2}\big(W(t) + \widehat{W}(T-t) - \widehat{W}(T)\big)$$

we obtain:

$$\begin{aligned}
&Q_n^{l,\eta}\big(\big(\mathcal{F}m(A_0, A_T, \varepsilon, \delta)\big) \cap \big(\mathbb{R}^d\big)^n\big) \\
&\quad = Q_n^{l,\eta}\left(\begin{array}{l}\exists i \leqslant n \text{ such that } X_i(0) \in A_0,\, X_i(T) \in A_T \text{ and} \\[1ex] \displaystyle\sup_{\substack{|t-s|<\delta \\ 0\leqslant s,t\leqslant T}} \big|W_i(t) - W_i(s) + \widehat{W}_i(T-t) - \widehat{W}_i(T-s)\big| > 2\varepsilon\end{array}\right) \\
&\quad \leqslant Q_n^{l,\eta}\big(\exists i \leqslant n \text{ such that } X_i(0) \in A_0 \text{ and } w(W_i, \delta, T) > \varepsilon\big) \\
&\qquad + Q_n^{l,\eta}\big(\exists i \leqslant n \text{ such that } X_i(0) \in A_T \text{ and } w(\widehat{W}_i, \delta, T) > \varepsilon\big) \\
&\quad \leqslant \sum_{i=1}^{n} \nu_n^{l,\eta}(x_i \in A_0)\, Q_n^{l,\eta}\big(w(W_i, \delta, T) > \varepsilon\big) \\
&\qquad + \sum_{i=1}^{n} \nu_n^{l,\eta}(x_i \in A_T)\, Q_n^{l,\eta}\big(w(W_i, \delta, T) > \varepsilon\big) \\
&\quad \leqslant n\, Q_n^{l,\eta}\big(w(W_1, \delta, T) > \varepsilon\big)\big(\nu_n^{l,\eta}(x_1 \in A_0) + \nu_n^{l,\eta}(x_1 \in A_T)\big).
\end{aligned}$$

By the scaling property of the Brownian Motion and Doob's inequality we have the upper bound:

$$Q_n^{l,\eta}\big(w(W_1, \delta, T) > \varepsilon\big) \leqslant 2\sqrt{5}\frac{T}{\delta}\exp\left(-\frac{\varepsilon^2}{20\delta}\right).$$

According to the definition (3) of $\beta_n^{l,\eta}$, since $\psi^{l,\eta} \geqslant 0$ and by Remark 2.1:

$$\beta_n^{l,\eta}(x_1, \dots, x_n) \geqslant 2\overline{N}\underline{\varphi} + \beta_{n-1}^{l,\eta}(x_2, \dots, x_n) \tag{5}$$

which implies that

$$
\begin{aligned}
(6)\qquad & \nu_n^{l,\eta}(x_1 \in A_0) \\
&= \int\limits_{(\mathbb{R}^d)^n} \mathbb{1}_{x_1\in A_0}\mathbb{1}_{\mathcal{A}}(x_1,\ldots,x_n)\,\mathrm{e}^{-\beta_n^{l,\eta}(x_1,\ldots,x_n)}\,dx_1\cdots dx_n \\
&\leqslant \mathrm{e}^{-2\overline{N}\underline{\varphi}}|A_0|\,\nu_{n-1}^{l,\eta}\big(\big(\mathbb{R}^d\big)^{n-1}\big)
\end{aligned}
$$

and the same result holds for A_T. This leads to the estimate:

$$
\begin{aligned}
& Q_n^{l,\eta}\big(\mathcal{F}m(A_0,A_T,\varepsilon,\delta)\cap\big(\mathbb{R}^d\big)^n\big) \\
&\quad\leqslant n\,\mathrm{e}^{-2\overline{N}\underline{\varphi}}\nu_{n-1}^{l,\eta}\big(\big(\mathbb{R}^d\big)^{n-1}\big)\big(|A_0|+|A_T|\big)2\sqrt{5}\frac{T}{\delta}\exp\left(-\frac{\varepsilon^2}{20\delta}\right),
\end{aligned}
$$

and summing this over n we obtain the desired result:

$$
\begin{aligned}
& Q^{l,\eta}\big(\mathcal{F}m(A_0,A_T,\varepsilon,\delta)\big) \\
&\quad= \frac{1}{Z^{l,\eta}}\sum_{n=1}^{+\infty}\frac{z^n}{n!}Q_n^{l,\eta}\big(\mathcal{F}m(A_0,A_T,\varepsilon,\delta)\cap\big(\mathbb{R}^d\big)^n\big) \\
&\quad\leqslant \frac{z}{Z^{l,\eta}}\,\mathrm{e}^{-2\overline{N}\underline{\varphi}}\big(|A_0|+|A_T|\big)2\sqrt{5}\frac{T}{\delta}\exp\left(-\frac{\varepsilon^2}{20\delta}\right) \\
&\qquad\times\sum_{n=1}^{+\infty}\frac{z^{n-1}}{(n-1)!}\nu_{n-1}^{l,\eta}\big(\big(\mathbb{R}^d\big)^{n-1}\big) \\
&\quad\leqslant C_0\big(|A_0|+|A_T|\big)\frac{1}{\delta}\exp\left(-\frac{\varepsilon^2}{20\delta}\right)
\end{aligned}
$$

with $C_0 = 2\sqrt{5}Tz\,\mathrm{e}^{-2\overline{N}\underline{\varphi}}$. □

Proof of Proposition 5.2. – For j in $\mathbb{N}$, let $a_j = K+\sqrt{(T\varepsilon^2/\delta)+20Tj}$. The sequence $(a_j)_j$ increases from $a_0 = K+\sqrt{T\varepsilon^2/\delta}$ to $+\infty$. By definition,

$$
\begin{aligned}
& Q^{l,\eta}(\mathcal{F}m(K,\varepsilon,\delta)) \\
&\quad= Q^{l,\eta}\big(\exists i,\ \big|X_i(0)\big|\leqslant K \text{ and } w(X_i,\delta,T)>\varepsilon\big) \\
&\quad\leqslant Q^{l,\eta}\big(\exists i,\ \big|X_i(0)\big|\leqslant K \text{ and } w(X_i,\delta,T)>\varepsilon \text{ and } \big|X_i(T)\big|\leqslant a_0\big) \\
&\qquad+\sum_{j=0}^{+\infty}Q^{l,\eta}\big(\exists i,\ \big|X_i(0)\big|\leqslant K \text{ and } a_j<\big|X_i(T)\big|<a_{j+1}\big)
\end{aligned}
$$

$$\leqslant Q^{l,\eta}\big(\exists i,\ |X_i(0)| \leqslant K,\ \ |X_i(T)| \leqslant a_0 \text{ and } w(X_i,\delta,T) > \varepsilon\big)$$
$$+\sum_{j=0}^{+\infty} Q^{l,\eta}\big(\exists i,\ \ |X_i(0)| \leqslant K,\ a_j < |X_i(T)| < a_{j+1} \text{ and}$$
$$w(X_i,T,T)^2 > \big(T\varepsilon^2/\delta\big) + 20Tj\big).$$

Using Lemma 5.1 and remarking that $1/T \leqslant 1/\delta$, we obtain:

$$\begin{aligned}
Q^{l,\eta}\big(\mathcal{F}m(K,\varepsilon,\delta)\big) &\leqslant C_0\big(K^d + (a_0)^d\big)\big|B(0,1)\big|\frac{1}{\delta}\exp\left(-\frac{\varepsilon^2}{20\delta}\right)\\
&\quad + C_0\sum_{j=0}^{+\infty}\big(K^d + (a_{j+1})^d - (a_j)^d\big)\big|B(0,1)\big|\frac{1}{T}\\
&\quad \times \exp\left(-\frac{\frac{T\varepsilon^2}{\delta} + 20Tj}{20T}\right)\\
&\leqslant \frac{C_0|B(0,1)|}{\delta}\,\mathrm{e}^{-\varepsilon^2/20\delta}\left(K^d + (a_0)^d + K^d\sum_{j=0}^{+\infty}\mathrm{e}^{-j}\right.\\
&\quad \left. + \sum_{j=0}^{+\infty}\big((a_{j+1})^d - (a_j)^d\big)\,\mathrm{e}^{-j}\right).
\end{aligned}$$

But $(a_{j+1})^d - (a_j)^d \leqslant 2^d\sqrt{20T}^d(a_j)^d$ and $a_j \leqslant 2K\sqrt{20Tj}\max(1, \sqrt{T\varepsilon^2/\delta})$, for $j \geqslant 1$. This leads to:

$$Q^{l,\eta}\big(\mathcal{F}m(K,\varepsilon,\delta)\big) \leqslant C^{st}\frac{1}{\delta}\exp\big(-\varepsilon^2/20\delta\big)K^d\max\Big(1,\sqrt{T\varepsilon^2/\delta}\Big)^d,$$

where C^{st} is a constant depending only on R, r, φ, d, T and z. This completes the proof. □

5.2. Probability of large chains

We introduce the notion of $(R+\varepsilon)$-chain of particles.

DEFINITION 5.3. – *Let $x \in \mathcal{A}$ and $\varepsilon > 0$. For each subset $\{x_1,\dots,x_n\}$ of x verifying $|x_1 - x_2| \leqslant R+\varepsilon,\dots,|x_{n-1} - x_n| \leqslant R+\varepsilon$, the set $\bigcup_{i=1}^n B(x_i, R+\varepsilon)$ is called an $(R+\varepsilon)$-chain of particles of x.*

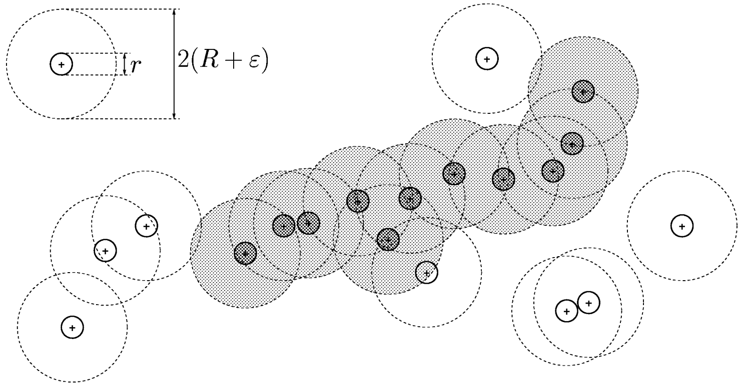

Fig. 1. Example of an $(R+\varepsilon)$-chain (pigmented in grey).

Now, let us fix $K \in \mathbb{N}^*$, $M \in \mathbb{N}^*$ and $\varepsilon > 0$ and let $Ch(K, M, R+\varepsilon)$ denote the event that there exists an $(R+\varepsilon)$-chain of M particles with one end inside of $B(0, K)$, that is:

$$Ch(K, M, R+\varepsilon) = \bigl\{x \in \mathcal{A},\ \exists\{x_1, \ldots, x_M\} \text{ subset of } x,\ |x_1| < K \\ \text{and } |x_1 - x_2| \leqslant R+\varepsilon, \ldots, |x_{M-1} - x_M| \leqslant R+\varepsilon\bigr\}.$$

Our aim here is to estimate the probability, under $\mu^{l,\eta}$, that such a chain exists.

PROPOSITION 5.4. – *For every $K \in \mathbb{N}^*$, $M \in \mathbb{N}^*$ and $\varepsilon > 0$, and for every $l \in \mathbb{N}^*$ and $\eta \in \mathcal{A}$, we have*:

$$\mu^{l,\eta}\bigl(Ch(K, M, R+\varepsilon)\bigr) \\ \leqslant \frac{1}{R^d - r^d} K^d \bigl(z\bigl|B(0,1)\bigr| \exp\bigl(-2\overline{N}\underline{\varphi}\bigr)\bigl((R+\varepsilon)^d - r^d\bigr)\bigr)^M.$$

From this proposition, we easily deduce the following corollary.

COROLLARY 5.5. – *There exists a critical activity*

$$z_c = \frac{\exp(2\overline{N}\underline{\varphi})}{(R^d - r^d)|B(0,1)|}$$

such that for $z < z_c$ one may find some constants $\varepsilon_0 \in\,]0,1[$, $C_3 \geqslant 0$ and $C_4 > 0$, only depending on z, R, r, φ and d, such that

$$\forall K, M, l \in \mathbb{N}^*,\ \forall \eta \in \mathcal{A},\ \forall \varepsilon \leqslant \varepsilon_0,$$
$$\mu^{l,\eta}\big(Ch(K, M, R+\varepsilon)\big) \leqslant C_3 K^d \,\mathrm{e}^{-C_4 M}.$$

Remark 5.6. – Proposition 5.4 still works for pure hard core interaction ($R = r$ and $\varphi \equiv 0$). In this case:

$$\mu^{l,\eta}\big(Ch(K, M, R+\varepsilon)\big) \leqslant K^d \big(z\big|B(0,1)\big|\big)^M \big((r+\varepsilon)^d - r^d\big)^{M-1}.$$

The condition on z disappears since $z_c = +\infty$. Our method is then an alternative elementary proof of Lemma 3.1 in [8], which was proved using powerful results of percolation theory.

Proof of Corollary 5.5. – If $z < z_c$ then one can choose ε_0 small enough to have $\mathrm{e}^{-C_4} = z|B(0,1)| \exp(-2\overline{N}\underline{\varphi})((R+\varepsilon_0)^d - r^d) < 1$. This gives the desired result with $C_3 = 1/(R^d - r^d)$. $\square$

Proof of Proposition 5.4. – By definition of $\mu^{l,\eta}$ (cf. (4)) we have

$$\begin{aligned} \mu^{l,\eta}&\big(Ch(K, M, R+\varepsilon)\big) \\ &= \frac{1}{Z^{l,\eta}} \sum_{n \geqslant M} \frac{z^n}{n!} \nu_n^{l,\eta}\big(\mathbb{R}^{dn} \cap Ch(K, M, R+\varepsilon)\big). \end{aligned} \tag{7}$$

But

$$\begin{aligned} &\nu_n^{l,\eta}\big(Ch(K, M, R+\varepsilon)\big) \\ &\quad = \frac{n!}{(n-M)!} \int\limits_{\mathcal{A}\cap\mathbb{R}^{dn}} \mathrm{e}^{-\beta_n^{l,\eta}(x_1,\ldots,x_n)} \mathbb{1}_{|x_1|<K} \prod_{i=1}^{M-1} \mathbb{1}_{|x_i - x_{i+1}| \leqslant R+\varepsilon}\, dx_1 \cdots dx_n. \end{aligned}$$

Iterating inequality (5) established in the proof of Lemma 5.1, we have:

$$\beta_n^{l,\eta}(x_1, \ldots, x_n) \geqslant 2M\overline{N}\underline{\varphi} + \beta_{n-M}^{l,\eta}(x_{M+1}, \ldots, x_n). \tag{8}$$

So we have:

$$\begin{aligned} &\nu_n^{l,\eta}\big(Ch(K, M, R+\varepsilon)\big) \\ &\quad \leqslant \frac{n!\,\mathrm{e}^{-2M\overline{N}\underline{\varphi}}}{(n-M)!} \int\limits_{\mathbb{R}^{dM}} \mathbb{1}_{|x_1|<K} \prod_{i=1}^{M-1} \mathbb{1}_{r \leqslant |x_i - x_{i+1}| \leqslant R+\varepsilon}\, dx_1 \cdots dx_M \\ &\qquad \times \nu_{n-M}^{l,\eta}\big(\mathbb{R}^{d(n-M)}\big) \end{aligned}$$

$$\leqslant \frac{n!\,\mathrm{e}^{-2M\overline{N}\underline{\varphi}}}{(n-M)!}\nu_{n-M}^{l,\eta}\big(\mathbb{R}^{d(n-M)}\big)\big(\big((R+\varepsilon)^d-r^d\big)\big|B(0,1)\big|\big)^{M-1}$$
$$\times K^d\big|B(0,1)\big|.$$

Introducing this in (7), we obtain:

$$\mu^{l,\eta}\big(Ch(K,M,R+\varepsilon)\big)$$
$$\leqslant \frac{z^M}{Z^{l,\eta}}\sum_{n\geqslant M}\frac{z^{n-M}}{(n-M)!}\nu_{n-M}^{l,\eta}\big(\mathbb{R}^{d(n-M)}\big)K^d\big|B(0,1)\big|^M$$
$$\times\,\mathrm{e}^{-2M\overline{N}\underline{\varphi}}\frac{((R+\varepsilon)^d-r^d)^M}{(R+\varepsilon)^d-r^d}$$
$$\leqslant \frac{1}{R^d-r^d}K^d\big(z\big|B(0,1)\big|\,\mathrm{e}^{-2\overline{N}\underline{\varphi}}\big((R+\varepsilon)^d-r^d\big)\big)^M. \quad \square$$

5.3. Probability of too high interaction between particles

Let $\mathcal{B}t(m,a)$ denote the following set of “bad trajectories”:

$$\mathcal{B}t(m,a)=\left\{X\in\mathcal{C}(\mathbb{R}^+,\mathcal{A}),\ \begin{array}{l}\exists i,\ w\big(X_i,\frac{1}{m},T\big)>\dfrac{\varepsilon_0}{4}\text{ and }\exists t\leqslant T,\\ |X_i(t)|\leqslant a+2m^2\\ \text{ or}\\ \exists k,\ j\in\{0,\dots,mT-1\},\text{ there exists}\\ \text{ an }(R+\varepsilon_0)\text{-chain of particles of}\\ X\big(\frac{k}{m}\big)\text{ which intersects both}\\ B\big(0,a+j\frac{m}{T}\big)\text{ and}\\ B\big(0,a+(j+1)\frac{m}{T}\big)^c\end{array}\right\},$$

where m and a are in $\mathbb{N}^*$ and ε_0 has been defined in Corollary 5.5.

PROPOSITION 5.7. – *If z is small enough ($z<z_c$), we have for $m,a\in\mathbb{N}^*$:*

$$\forall l\in\mathbb{N}^*,\ \forall\eta\in\mathcal{A},\qquad Q^{l,\eta}\big(\mathcal{B}t(m,a)\big)\leqslant C_5a^d\,\mathrm{e}^{-C_6m},$$

where $C_5\geqslant 0$ and $C_6>0$ depend only on R, r, φ, d, T and z.

Proof of Proposition 5.7. –

$$
\begin{aligned}
&Q^{l,\eta}\left(\exists i, w\left(X_i, \frac{1}{m}, T\right) > \frac{\varepsilon_0}{4} \text{ and } \exists t \leqslant T, \left|X_i(t)\right| \leqslant a + 2m^2\right)\\
&\quad\leqslant Q^{l,\eta}\left(\exists i,\ w\left(X_i, \frac{1}{m}, T\right) > \frac{\varepsilon_0}{4} \text{ and } \left|X_i(0)\right| \leqslant a + 3m^2\right)\\
&\qquad + Q^{l,\eta}\left(\exists i,\ \left|X_i(0)\right| > a + 3m^2 \text{ and } \exists t \leqslant T,\ \left|X_i(t)\right| \leqslant a + 2m^2\right).
\end{aligned}
$$

But the second term of this bound is smaller than

$$
\begin{aligned}
\sum_{j=1}^{+\infty} Q^{l,\eta}\big(\exists i, a + (2+j)m^2 < \left|X_i(0)\right| \leqslant a + (3+j)m^2 \text{ and}\\
w(X_i, T, T) > jm^2\big).
\end{aligned}
$$

Thus using Proposition 5.2, we obtain the new upper bound:

$$
\begin{aligned}
&Q^{l,\eta}\left(\exists i,\ w\left(X_i, \frac{1}{m}, T\right) > \frac{\varepsilon_0}{4} \text{ and } \exists t \leqslant T,\ \left|X_i(t)\right| \leqslant a + 2m^2\right)\\
&\quad\leqslant C_1 m\left(a + 3m^2\right)^d \exp\left(-\frac{C_2\varepsilon_0^2 m}{16}\right)\\
&\qquad + \sum_{j=1}^{+\infty} \frac{C_1}{T}\left(a + (3+j)m^2\right)^d \exp\left(-\frac{C_2 j^2 m^4}{T}\right)\\
&\quad\leqslant C_5' a^d\, \mathrm{e}^{-C_6' m}
\end{aligned}
$$

for C_5' and C_6' well chosen, depending only on R, r, φ, d, T and z.

We now have to bound

$$
Q^{l,\eta}\left(\exists k, j \in \{0, \ldots, mT-1\},\ \begin{array}{l}\text{there exists an } (R+\varepsilon_0)\text{-chain of}\\ \text{particles of } X\left(\frac{k}{m}\right) \text{ which intersects}\\ \text{both } B\left(0, a + j\frac{m}{T}\right)\\ \text{ and } B\left(0, a + (j+1)\frac{m}{T}\right)^c\end{array}\right).
$$

Thanks to the stationarity of $Q^{l,\eta}$, this probability is smaller than

$$\leqslant \sum_{j=0}^{mT-1}\sum_{k=0}^{mT-1}\mu^{l,\eta}\left(x\in\mathcal{A},\ \begin{array}{l}\text{there exists an } (R+\varepsilon_0)\text{-chain of particles}\\ \text{of } x \text{ which intersects both}\\ B\big(0,a+j\frac{m}{T}\big) \text{ and } B\big(0,a+(j+1)\frac{m}{T}\big)^c\end{array}\right)$$

$$\leqslant \sum_{j=0}^{mT-1} mT\mu^{l,\eta}\left(Ch\left(a+j\frac{m}{T}+R+\varepsilon_0,\left[\frac{m}{T(R+\varepsilon_0)}\right],R+\varepsilon_0\right)\right).$$

Because of Corollary 5.5, this is bounded by

$$\leqslant (mT)^2C_3\big(a+m^2+R+\varepsilon_0\big)^d\exp\left(-C_4\left(\frac{m}{T(R+\varepsilon_0)}-1\right)\right)$$
$$\leqslant C_5''a^d\,\mathrm{e}^{-C_6''m}$$

for some C_5'', C_6'' depending only on C_3, C_4, R, ε_0 and T. This completes the proof with $C_5=\max(C_5',C_5'')$ and $C_6=\min(C_6',C_6'')$. □

6. Convergence of the approximations

The aim of this section is to prove the convergence of the sequence $(X^l)_l$ to a limit process X^∞. We shall check in the next section that X^∞ is a solution of $(\mathcal{E})$.

Through this whole section, μ will denote a fixed element of $\mathcal{G}(z)$ with $z<z_c$, and we also fix a μ-distributed random variable $X^\infty(0)$ on $(\Omega,\mathcal{F},P)$. As we will prove in Proposition 7.5, this means that we first construct the process X^∞ in the reversible situation.

For each $l\in\mathbb{N}^*$, X^l denotes the solution of $(\mathcal{E}^l)$ on $(\Omega,\mathcal{F},P)$ verifying $X^l(0)=X^\infty(0)$. As usual for infinite-dimensional stochastic equations, we define a set $\Omega_0\subset\Omega$ of full measure and study $(\mathcal{E}^l)$ only for $\omega\in\Omega_0$.

For each ρ, $T\in\mathbb{N}^*$ and $l\geqslant\rho+1$, let $m(\rho,l,T)$ and $a(\rho,l,T)$ be integers defined by:

$$m(\rho,l,T)=\left[\sqrt{\frac{l-R-\rho-1}{T+1}}\right]\quad\text{and}\quad a(\rho,l,T)=\rho+m(\rho,l,T)T.$$

They verify

$$(9)\qquad \begin{cases} a(\rho,l,T) \geqslant \rho + \varepsilon_0 m(\rho,l,T)T, \\ a(\rho,l,T) + m(\rho,l,T)^2 + 1 < l - R \quad \text{and} \\ \sum_l a(\rho,l,T)^d \, \mathrm{e}^{-C_6 m(\rho,l,T)} < +\infty \end{cases}$$

(recall that ε_0 has been defined in Corollary 5.5).

Let

$$\begin{aligned} \Omega_0 = \big\{\omega \in \Omega \text{ such that, } &\forall \rho \in \mathbb{N}^*,\ \forall T \in \mathbb{N}^*,\ \exists l_0 \in \mathbb{N}^*,\ \forall l \geqslant l_0, \\ &X^l(\omega,\cdot) \notin \mathcal{B}t\big(m(\rho,l,T), a(\rho,l,T)\big) \text{ and} \\ &X^{l+1}(\omega,\cdot) \notin \mathcal{B}t\big(m(\rho,l,T), a(\rho,l,T)\big)\big\}. \end{aligned}$$

PROPOSITION 6.1. –

(i) *For every ω in Ω_0, every $T \in \mathbb{N}^*$ and every $i \in \mathbb{N}$, the sequence $(X_i^l(\omega,t),\ L_{ij}^l(\omega,t),\ j \in \mathbb{N}, t \in [0,T])_{l \in \mathbb{N}^*}$ of elements of $\mathcal{C}([0,T], \mathbb{R}^d \times \mathbb{R}_+^{\mathbb{N}})$ converges in the sense of uniform convergence of continuous paths to a limit denoted by $(X_i^\infty(\omega,t), L_{ij}^\infty(\omega,t),$ $j \in \mathbb{N},\ t \in [0,T])$.*
Moreover, this sequence is stationary:

$$\begin{aligned} &\forall \omega \in \Omega_0,\ \forall T \in \mathbb{N}^*,\ \forall \rho \in \mathbb{N}^*,\ \forall i \text{ such that } \big|X_i^\infty(\omega,0)\big| \leqslant \rho, \\ &\exists l_0,\ \forall l \geqslant l_0,\ X_i^l(\omega,\cdot) = X_i^\infty(\omega,\cdot) \end{aligned}$$

and

$$\forall j \in \mathbb{N},\ L_{ij}^l(\omega,\cdot) = L_{ij}^\infty(\omega,\cdot) \text{ on } [0,T].$$

(ii) *Furthermore, the convergence holds in $\mathcal{C}([0,T], \mathcal{M})$ for each $T \in \mathbb{N}^*$:*

$$\forall \omega \in \Omega_0,\ X^\infty(\omega,\cdot) = \lim_{l \to +\infty} X^l(\omega,\cdot) \text{ on } [0,T].$$

(iii) *$P(\Omega_0) = 1$ and then the sequence of processes $(X^l)_{l \in \mathbb{N}^*} \in \mathcal{C}(\mathbb{R}^+, \mathcal{M})$ converges in distribution to the process $X^\infty \in \mathcal{C}(\mathbb{R}^+, \mathcal{A})$.*

Proof of Proposition 6.1. – We first prove that Ω_0 is of full measure:

$$P(\Omega-\Omega_0)\leqslant\sum_{\rho,T\in\mathbb{N}^*}P\biggl(\bigcap_{l_0\in\mathbb{N}^*}\bigcup_{l\geqslant l_0}\bigl\{X^l\in\mathcal{B}t\bigl(m(\rho,l,T),a(\rho,l,T)\bigr)\\ \cup\mathcal{B}t\bigl(m(\rho,l-1,T),a(\rho,l-1,T)\bigr)\bigr\}\biggr).$$

According to Borel–Cantelli lemma, it is enough to prove that, for $\rho,T\in\mathbb{N}^*$

$$\sum_{l=1}^{+\infty}P\bigl(X^l\in\mathcal{B}t\bigl(m(\rho,l,T),a(\rho,l,T)\bigr)\\ \cup\mathcal{B}t\bigl(m(\rho,l-1,T),a(\rho,l-1,T)\bigr)\bigr)<+\infty. \tag{10}$$

From now on, let us fix ρ, T and l and simply denote by $\mathcal{B}t$ the set

$$\mathcal{B}t\bigl(m(\rho,l,T),a(\rho,l,T)\bigr)\cup\mathcal{B}t\bigl(m(\rho,l-1,T),a(\rho,l-1,T)\bigr).$$

We shall show (step 1) that

$$P\bigl(X^l\in\mathcal{B}t\bigr)\leqslant\int_{\mathcal{A}}Q^{l,\eta}(X\in\mathcal{B}t)\,d\mu(\eta)+2\int_{\mathcal{A}}\biggl(1-\frac{Z^{\Lambda,\eta}}{Z^{l,\eta}}\biggr)\,d\mu(\eta) \tag{11}$$

and (step 2) that, for $\eta\in\mathcal{A}$,

$$0\leqslant 1-\frac{Z^{\Lambda,\eta}}{Z^{l,\eta}}\leqslant z\exp\bigl(-2\overline{N}\underline{\varphi}\bigr)\int_{\mathbb{R}^d}\mathbb{1}_{\psi^{l,\eta}(x)>0}\exp\bigl(-\psi^{l,\eta}(x)\bigr)\,dx. \tag{12}$$

This two inequalities imply the summability (10) (use (9), assumption (2) on $\psi^{l,\eta}$ and Proposition 5.7 about the exponential decrease of $Q^{l,\eta}(X\in\mathcal{B}t)$ uniformly in η).

Step 1. *Proof of* (11).

$$\begin{aligned}P\bigl(X^l\in\mathcal{B}t\bigr)&=\int_{\mathcal{A}}E\bigl(\mathbb{1}_{X^l\in\mathcal{B}t}|X^l(0)=\eta\bigr)\,d\mu(\eta)\\ &=\int_{\mathcal{A}}\int_{\mathcal{A}}E\bigl(\mathbb{1}_{X^l\in\mathcal{B}t}|X^l(0)=\xi_\Lambda\eta_{\Lambda^c}\bigr)\,d\mu(\xi|\mathcal{B}_{\Lambda^c})(\eta)\,d\mu(\eta).\end{aligned}$$

By definition of X^l and $X^{l,\eta}$, for $\eta, \xi \in \mathcal{A}$ such that $N_{\Lambda^c}(\xi) = 0$:

$$0 \leqslant E\big(\mathbb{1}_{X^l \in \mathcal{B}t} \mid X^l(0) = \xi_\Lambda \eta_{\Lambda^c}\big) = E\big(\mathbb{1}_{X^{l,\eta} \in \mathcal{B}t} \mid X^{l,\eta}(0) = \xi\big) \leqslant 1.$$

These inequalities imply that:

$$\begin{aligned} P\big(X^l \in \mathcal{B}t\big) \leqslant &\int_{\mathcal{A}} Q^{l,\eta}(X \in \mathcal{B}t)\, d\mu(\eta) \\ &+ \int_{\mathcal{A}} \Big(\big|\mu^{l,\eta} - \mu(\cdot|\mathcal{B}_{\Lambda^c})(\eta)\big|(N_{\Lambda^c \cup B(\eta \cap \Lambda^c, r)} = 0) \\ &+ \mu^{l,\eta}(N_{\Lambda^c \cup B(\eta_{\Lambda^c}, r)} \geqslant 1)\Big) d\mu(\eta). \end{aligned}$$

But

$$\begin{aligned} &\big|\mu^{l,\eta} - \mu(\cdot \mid \mathcal{B}_{\Lambda^c})(\eta)\big|(N_{\Lambda^c \cup B(\eta_{\Lambda^c}, r)} = 0) \\ &\quad = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \left(\prod_{i=1}^n \mathbb{1}_{\xi_i \in \Lambda} \mathbb{1}_{d(\eta \cap \Lambda^c, \xi_i) \geqslant r}\right) \mathbb{1}_{\mathcal{A}}(\xi_1, \ldots, \xi_n) \\ &\qquad \times \left|\frac{e^{-\beta_n^{l,\eta}(\xi_1,\ldots,\xi_n)}}{Z^{l,\eta}} - \frac{e^{-\beta_n^{l,\eta}(\xi_1,\ldots,\xi_n)}}{Z^{\Lambda,\eta}}\right| d\xi_1 \cdots d\xi_n \\ &\quad = \left|\frac{1}{Z^{l,\eta}} - \frac{1}{Z^{\Lambda,\eta}}\right| \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} \mathbb{1}_{\mathcal{A}}(\xi_\Lambda \eta_{\Lambda^c})\, e^{-\beta_n^{l,\eta}(\xi_1,\ldots,\xi_n)}\, d\xi_1 \cdots d\xi_n \\ &\quad = \left|\frac{Z^{\Lambda,\eta}}{Z^{l,\eta}} - 1\right| \end{aligned}$$

and similarly

$$\begin{aligned} &0 \leqslant \mu^{l,\eta}(N_{\Lambda^c \cup B(\eta_{\Lambda^c}, r)} \geqslant 1) \\ &\quad = 1 - \frac{1}{Z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \left(\prod_{i=1}^n \mathbb{1}_{\xi_i \in \Lambda} \mathbb{1}_{d(\eta \cap \Lambda^c, \xi_i) \geqslant r}\right) \mathbb{1}_{\mathcal{A}}(\xi_1, \ldots, \xi_n) \\ &\qquad \times e^{-\beta_n^{l,\eta}(\xi_1,\ldots,\xi_n)}\, d\xi_1 \cdots d\xi_n = 1 - \frac{Z^{\Lambda,\eta}}{Z^{l,\eta}}. \end{aligned}$$

Step 2. *Proof of* (12).

$$\frac{1}{Z^{l,\eta}}\big(Z^{l,\eta} - Z^{\Lambda,\eta}\big) =$$

$$
\begin{aligned}
&= \frac{1}{Z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int\limits_{\mathcal{A}\cap\mathbb{R}^{dn}} \mathrm{e}^{-\beta_n^{l,\eta}(\xi_1,\dots,\xi_n)} \left(1 - \prod_{i=1}^{n} \mathbb{1}_{\Lambda - B(\eta_{\Lambda^c},r)}(\xi_i)\right) d\xi_1 \cdots d\xi_n \\
&\leqslant \frac{1}{Z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int\limits_{\mathcal{A}\cap\mathbb{R}^{dn}} \mathrm{e}^{-\beta_n^{l,\eta}(\xi_1,\dots,\xi_n)} \left(\sum_{i=1}^{n} \mathbb{1}_{\psi^{l,\eta}(\xi_i)>0}\right) d\xi_1 \cdots d\xi_n \\
&\leqslant \frac{1}{Z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} n \int\limits_{\mathcal{A}\cap\mathbb{R}^{dn}} \mathrm{e}^{-\psi^{l,\eta}(\xi_1)}\, \mathrm{e}^{-2\overline{N}\underline{\varphi} - \beta_{n-1}^{l,\eta}(\xi_2,\dots,\xi_n)} \mathbb{1}_{\psi^{l,\eta}(\xi_1)>0}\, d\xi_1 \cdots d\xi_n \\
&\leqslant z \exp\bigl(-2\overline{N}\underline{\varphi}\bigr) \int\limits_{\mathbb{R}^d} \mathbb{1}_{\psi^{l,\eta}(x)>0} \exp\bigl(-\psi^{l,\eta}(x)\bigr)\, dx.
\end{aligned}
$$

Proof of Proposition 6.1(i). – In this whole proof, $\omega \in \Omega_0$, $T \in \mathbb{N}^*$ and $\rho \in \mathbb{N}^*$ are fixed, and we also fix an $l > l_0$ (which appears in the definition of Ω_0). Consequently, $m(\rho, l, T)$ and $a(\rho, l, T)$ are fixed too, and will simply be denoted by m and a.

For k between 0 and $mT - 1$, let $C_k^l(\omega)$ denote the union of $B(0, a + m^2 - (k+1)(m/T))$ with the $(R + \varepsilon_0)$-chains of particles of $X^l(\omega, k/m)$ which intersect $B(0, a + m^2 - k(m/T))^c$.

Since $\omega \in \Omega_0$, $C_k^l(\omega) \subset B(0, a + m^2 - k(m/T))$. We also introduce $J_k^l(\omega) = \{i \in \mathbb{N}$ such that $X_i^l(\omega, k/m) \in C_k^l(\omega)\}$, which by definition verifies:

$$
\forall i \in J_k^l(\omega),\ \forall j \notin J_k^l(\omega), \qquad \left| X_i^l\left(\omega, \frac{k}{m}\right) - X_j^l\left(\omega, \frac{k}{m}\right) \right| > R + \varepsilon_0
$$

and: $\forall i \in J_k^l(\omega),\ |X_i^l(\omega, (k/m))| \leqslant a + m^2 - k(m/T)$.

Since no particle of $X^l(\omega, \cdot)$ in $B(0, a + m^2)$ moves for more than $\varepsilon_0/4$ during a time period of length $1/m$, one has in fact

$$
\begin{aligned}
\text{(13)} \qquad &\forall i \in J_k^l(\omega),\ \forall j \notin J_k^l(\omega),\ \forall t \in \left[\frac{k}{m}, \frac{k+1}{m}\right], \\
&\bigl|X_i^l(\omega, t) - X_j^l(\omega, t)\bigr| > R + \frac{\varepsilon_0}{2}
\end{aligned}
$$

and (recall (9))

$$
\forall i \in J_k^l(\omega),\ \forall t \in \left[\frac{k}{m}, \frac{k+1}{m}\right],
$$

$$\left|X_i^l(\omega,t)\right| \leqslant a+m^2-k\frac{m}{T}+\frac{\varepsilon_0}{4} \leqslant l-R$$

which implies that no interaction is possible during the time interval $[\frac{k}{m},\frac{k+1}{m}]$ between the particles of $J_k^l(\omega)$ and the other particles, and that $\psi^{l,X^l(0)}(X_i^l(\omega,t))$ vanishes for t in $[\frac{k}{m},\frac{k+1}{m}]$. Remark that

$$j \notin J_k^l(\omega) \Rightarrow \left|X_j^l\left(\omega,\frac{k}{m}\right)\right| > a+m^2-(k+1)\frac{m}{T}+(R+\varepsilon_0)$$

and that the same argument of "slow motion" implies that for each k in $\{0,\dots,mT-1\}$,

$$\begin{aligned} j \notin J_k^l(\omega) &\Rightarrow \left|X_j^l\left(\omega,\frac{k+1}{m}\right)\right| > a+m^2-(k+1)\frac{m}{T}+R+\frac{3}{4}\varepsilon_0 \\ &\Rightarrow X_j^l\left(\omega,\frac{k+1}{m}\right) \notin C_{k+1}^l(\omega) \\ &\qquad\qquad \subset B\left(0,a+m^2-(k+1)\frac{m}{T}\right) \\ &\Rightarrow j \notin J_{k+1}^l(\omega). \end{aligned} \tag{14}$$

In this context, the equation $(\mathcal{E}^l)$ verified by $X^l(\omega,t)$ for $t\in[\frac{k}{m},\frac{k+1}{m}]$ reduces to the following equation called $(\mathcal{E}(k,J_k^l(\omega)))$:

$$\begin{aligned} X_i^l(\omega,t) = {} & X_i^l\left(\omega,\frac{k}{m}\right)+W_i(\omega,t)-W_i\left(\omega,\frac{k}{m}\right) \\ & -\frac{1}{2}\int_{\frac{k}{m}}^{t}\sum_{j\in J_k^l(\omega)}\nabla\varphi\big(X_i^l-X_j^l\big)(\omega,s)\,ds \\ & +\int_{\frac{k}{m}}^{t}\sum_{j\in J_k^l(\omega)}\big(X_i^l(\omega,s)-X_j^l(\omega,s)\big)\,dL_{ij}^l(\omega,s), \\ & i\in J_k^l(\omega). \end{aligned} \tag{15}$$

Now remark that all that has been done for $X^l(\omega,\cdot)$ can be done for $X^{l+1}(\omega,\cdot)$. And we similarly obtain that

$$\left(X_i^{l+1}(\omega,t),\, t\in\left[\frac{k}{m},\frac{k+1}{m}\right]\right)_{i\in J_k^{l+1}(\omega)}$$

is the solution of the equation $(\mathcal{E}(k, J_k^{l+1}(\omega)))$.

But since $X^l(\omega,0) = X^{l+1}(\omega,0) = X^\infty(\omega,0)$ and $L^l(\omega,0) = L^{l+1}(\omega, 0) = 0$, the sets $J_0^l(\omega)$ and $J_0^{l+1}(\omega)$ are equal and $(\mathcal{E}(0, J_0^l(\omega)))$ is the same equation as $(\mathcal{E}(0, J_0^{l+1}(\omega)))$. Then for $t \in [0, \frac{1}{m}]$:

$$\forall i, j \in J_0^l(\omega) = J_0^{l+1}(\omega), \quad X_i^l(\omega,t) = X_i^{l+1}(\omega,t)$$

and

$$L_{ij}^l(\omega,t) = L_{ij}^{l+1}(\omega,t)$$

and because $J_1^l(\omega) \subset J_0^l(\omega)$ (and idem for $J_1^{l+1}(\omega)$) this in turn implies that $J_1^l(\omega) = J_1^{l+1}(\omega)$.

By induction, we thus obtain that

$$\begin{aligned}
&\forall k \in \{1,\dots,mT-1\},\ J_k^l(\omega) = J_k^{l+1}(\omega) \quad \text{and}\\
&\forall k \in \{1,\dots,mT-1\},\ \forall i,j \in J_k^l(\omega),\ \forall t \in [0,(k+1)/m]\\
&X_i^l(\omega,t) = X_i^{l+1}(\omega,t) \quad \text{and} \quad L_{ij}^l(\omega,t) = L_{ij}^{l+1}(\omega,t).
\end{aligned}$$

This means that $X_i^l(\omega,\cdot)$ and $X_i^{l+1}(\omega,\cdot)$ are equal on $[0,T]$ for $i \in J_{mT-1}^l(\omega)$, and the same result holds for $(L_{ij}^l(\omega,\cdot))_{i,j}$ and $(L_{ij}^{l+1}(\omega,\cdot))_{i,j}$ because $L_{ij}^l(\omega,\cdot)$ and $L_{ij}^{l+1}(\omega,\cdot)$ identically vanish for $i \in J_{mT-1}^l(\omega)$ and $j \notin J_{mT-1}^l(\omega)$ (recalling (13)).

Now, using once more the “slow motion” property of X^l and (9), we see that

$$\begin{aligned}
|X_i^\infty(\omega,0)| \leqslant \rho \Rightarrow X_i^l(\omega,T) &\in B\left(0, \rho + \frac{\varepsilon_0}{4} mT\right)\\
&\subset B(0, \rho + \varepsilon_0 mT) \subset C_{mT-1}^l(\omega)\\
&\Rightarrow i \in J_{mT-1}^l(\omega)
\end{aligned} \tag{16}$$

and since ρ may be chosen arbitrary large, Proposition 6.1(i) is proven. □

Proof of Propositions 6.1(ii) and (iii). – Recall that $\mathcal{M}$ is endowed with the vague topology. Then the convergence of $(X^l(\omega,\cdot))_l$ takes place in $\mathcal{C}([0,T],\mathcal{M})$ if and only if, for $f \in \mathcal{C}_c(\mathbb{R}^d)$,

$$\sum_i f\left(X_i^l(\omega,t)\right) \underset{l\to+\infty}{\longrightarrow} \sum_i f\left(X_i^\infty(\omega,t)\right) \quad \text{uniformly in } t \in [0,T].$$

Since f has a compact support, all the terms in the above sum vanish except at most for a finite number of indices. Thus the convergence follows directly from Proposition 6.1(i).

The convergence in $\mathcal{C}(\mathbb{R}^+, \mathcal{M})$ is defined as the convergence in $\mathcal{C}([0,T], \mathcal{M})$ for each $T \in \mathbb{N}^*$, which is proven in Proposition 6.1(ii). □

7. Proofs of the main results

Main Theorems 3.2 and 3.3 are now consequences of Propositions 7.1, 7.4 and 7.5.

PROPOSITION 7.1. – *The process* $(X_i^\infty(t), L_{ij}^\infty(t),\ i, j \in \mathbb{N},\ t \in \mathbb{R}^+)$ *defined in Proposition* 6.1 *is a solution of equation* $(\mathcal{E})$.

For $T \in \mathbb{N}^*$ fixed and for $\tilde{m}, \tilde{a} \in \mathbb{N}^*$, let $\tilde{\mathcal{B}}t(\tilde{m}, \tilde{a})$ denote the following subset of $\mathcal{B}t(\tilde{m}, \tilde{a})$ (defined in Section 5.3):

$$\tilde{\mathcal{B}}t(\tilde{m}, \tilde{a}) = \left\{ X \in \mathcal{C}\big(\mathbb{R}^+, \mathcal{A}\big), \exists i,\ w\left(X_i, \frac{1}{\tilde{m}}, T\right) > \frac{\varepsilon_0}{4} \text{ and } \exists t \leqslant T, \right.$$
$$\left. \big|X_i(t)\big| \leqslant \tilde{a} + 2\tilde{m}^2 \right\}$$

and

$$\Omega(\tilde{m}, \tilde{a}) = \limsup_{l \to +\infty} \big\{\omega \in \Omega_0,\ X^l(\omega, \cdot) \notin \tilde{\mathcal{B}}t(\tilde{m}, \tilde{a})\big\}.$$

LEMMA 7.2. – *For each* $\tilde{m}, \tilde{a} \in \mathbb{N}^*$ *one has* $P(\Omega(\tilde{m}, \tilde{a})^c) \leqslant C_5 \tilde{a}^d \times \mathrm{e}^{-C_6 \tilde{m}}$ *where* $C_5 \geqslant 0$ *and* $C_6 > 0$ *are the constants defined in Proposition* 5.7.

LEMMA 7.3. – *For each* $\tilde{m}, \tilde{a} \in \mathbb{N}^*$ *one also has*

$$\forall \omega \in \Omega(\tilde{m}, \tilde{a}), \quad X^\infty(\omega, \cdot) \notin \tilde{\mathcal{B}}t(\tilde{m}, \tilde{a}).$$

Proof of Lemma 7.2. – By definition of $\Omega(\tilde{m}, \tilde{a})$ and thanks to Fatou lemma

$$\begin{aligned} P\big(\Omega(\tilde{m}, \tilde{a})^c\big) &\leqslant \liminf_{l \to +\infty} P\big(X^l(\omega, \cdot) \in \tilde{\mathcal{B}}t(\tilde{m}, \tilde{a})\big) \\ &\leqslant \liminf_{l \to +\infty} P\big(X^l(\omega, \cdot) \in \mathcal{B}t(\tilde{m}, \tilde{a})\big). \end{aligned}$$

Exactly the same computation as the one proving inequality (11) gives:

$$
\begin{aligned}
&P\bigl(X^l \in \mathcal{B}t(\tilde{m},\tilde{a})\bigr) \\
&\quad\leqslant \int_{\mathcal{A}} Q^{l,\eta}\bigl(X \in \mathcal{B}t(\tilde{m},\tilde{a})\bigr)\,d\mu(\eta) + 2\int_{\mathcal{A}} \left(1 - \frac{Z^{\Lambda,\eta}}{Z^{l,\eta}}\right) d\mu(\eta) \\
&\quad\leqslant C_5\tilde{a}^d\,\mathrm{e}^{-C_6\tilde{m}} + z\,\mathrm{e}^{-2\overline{N}\underline{\varphi}} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{l,\eta}(x)>0} \exp\bigl(-\psi^{l,\eta}(x)\bigr)\,dx
\end{aligned}
$$

by using Proposition 5.7 and inequality (12), which completes the proof. □

Proof of Lemma 7.3. – According to Proposition 6.1(i)

$$
\begin{aligned}
&\forall \omega \in \Omega_0,\ \forall i \in \mathbb{N},\ \exists l_0 \in \mathbb{N},\ \forall l \geqslant l_0, \\
&X_i^\infty(\omega,\cdot) = X_i^l(\omega,\cdot) \text{ on } [0,T].
\end{aligned}
$$

Consequently, for $\omega \in \Omega(\tilde{m},\tilde{a})$ and for $i \in \mathbb{N}$, there exists an $l \in \mathbb{N}$ such that

$$
\begin{aligned}
&X_i^\infty(\omega,\cdot) = X_i^l(\omega,\cdot) \text{ on } [0,T] \quad \text{and} \\
&\left(w\left(X_i^l(\omega,\cdot), \frac{1}{\tilde{m}}, T\right) \leqslant \frac{\varepsilon_0}{4} \quad \text{or} \quad \bigl|X_i^l(\omega,.)\bigr| > \tilde{a} + 2\tilde{m}^2\right),
\end{aligned}
$$

and thus $w(X_i^\infty(\omega,\cdot), 1/\tilde{m}, T) \leqslant \varepsilon_0/4$ or $\forall t \leqslant T,\ |X_i^\infty(\omega,t)| > \tilde{a} + 2\tilde{m}^2$. □

Proof of Proposition 7.1. –
Step 1.
For the first part of this proof, $T \in \mathbb{N}^*$, $\rho \in \mathbb{N}^*$ are fixed; we choose $\tilde{m} = \rho$ and $\tilde{a} = [\rho + \frac{\varepsilon_0}{4}\rho T + R]$, and we fix $\omega \in \Omega(\rho,\tilde{a})$. We freely use here the notations and results of Proposition 6.1(i). In particular, in the equation (15) the sum over $j \in J_k^l(\omega)$ of the interactions with the ith coordinate may be replaced by a sum over $j \in \mathbb{N}$.
Recalling (14), the assertion (16) implies that:

$$
\forall k \in \{0,\ldots,mT-1\}, \quad \bigl\{i \in \mathbb{N} \text{ such that } \bigl|X_i^\infty(\omega,0)\bigr| \leqslant \rho\bigr\} \subset J_k^l(\omega).
$$

We then obtain that for i such that $|X_i^\infty(\omega,0)| \leqslant \rho$ and $t \in [0,T]$,

$$X_i^l(\omega,t) = X_i^\infty(\omega,0) + W_i(\omega,t) - \frac{1}{2}\int_0^t \sum_{j\in\mathbb{N}} \nabla\varphi\big(X_i^l(\omega,s) - X_j^l(\omega,s)\big)\,ds + \int_0^t \sum_{j\in\mathbb{N}} \big(X_i^l(\omega,s) - X_j^l(\omega,s)\big)dL_{ij}^l(\omega,s). \tag{17}$$

Remark that $\tilde{a}$ verifies $\tilde{a} + 2\rho^2 \geqslant \rho + \frac{\varepsilon_0}{4}\rho T + R$ so for l's such that $X^l(\omega,\cdot) \notin \tilde{\mathcal{B}}t(\rho,\tilde{a})$ we have:

$$\begin{aligned} &\forall i \in \mathbb{N}, \quad \big|X_i^\infty(\omega,0)\big| \leqslant \rho \\ &\quad \Rightarrow \forall t \in [0,T], \quad \big|X_i^l(\omega,t)\big| \leqslant \rho + \frac{\varepsilon_0}{4}\rho T, \\ &\forall j \in \mathbb{N}, \quad \big|X_j^\infty(\omega,0)\big| > \rho + \frac{\varepsilon_0}{2}\rho T + R \\ &\quad \Rightarrow \forall t \leqslant T, \quad \big|X_j^l(\omega,t)\big| > \rho + \frac{\varepsilon_0}{4}\rho T + R. \end{aligned} \tag{18}$$

Since $\omega \in \Omega(\rho,\tilde{a})$, there exists an infinite number of indices l such that $X^l(\omega,\cdot) \notin \tilde{\mathcal{B}}t(\rho,\tilde{a})$ and equation (17) holds for this indices l. Eq. (17) still holds if we replace the sums over $j \in \mathbb{N}$ by sums over $\{j, |X_j^\infty(\omega,0)| \leqslant \rho + \frac{\varepsilon_0}{2}\rho T + R\}$, due to (18).

For this infinite number of l's, for all i such that $|X_i^\infty(\omega,0)| \leqslant \rho$, for $t \in [0,T]$,

$$X_i^l(\omega,t) = X_i^l(\omega,0) + W_i(\omega,t) - \frac{1}{2}\int_0^t \sum_{j,\ |X_j^\infty(\omega,0)|\leqslant\rho+\frac{\varepsilon_0}{2}\rho T+R} \nabla\varphi\big(X_i^l(\omega,s) - X_j^l(\omega,s)\big)\,ds + \int_0^t \sum_{j,\ |X_j^\infty(\omega,0)|\leqslant\rho+\frac{\varepsilon_0}{2}\rho T+R} \big(X_i^l(\omega,s) - X_j^l(\omega,s)\big)\,dL_{ij}^l(\omega,s). \tag{19}$$

We can choose an l large enough such that (19) holds and:

$$\forall j \text{ such that } \big|X_j^\infty(\omega,0)\big| \leqslant \rho + \frac{\varepsilon_0}{2}\rho T + R,$$
$$X_j^l(\omega,\cdot) = X_j^\infty(\omega,\cdot) \text{ on } [0,T].$$

Consequently (19) holds also with X^∞ instead of X^l. Now use Lemma 7.3: $X^\infty(\omega,\cdot) \notin \tilde{\mathcal{B}t}(\rho,\tilde{a})$ because $\omega \in \Omega(\rho,\tilde{a})$. As already remarked it is then equivalent to sum over $j \in \mathbb{N}$ or over $\{j, |X_j^\infty(\omega,0)| \leqslant \rho + \frac{\varepsilon_0}{2}\rho T + R\}$.

The final result of this step is thus: $\forall \rho \in \mathbb{N}^*,\ \forall \omega \in \Omega(\rho, [\rho + \frac{\varepsilon_0}{4}\rho T + R]),\ \forall i$ such that $|X_i^\infty(\omega,0)| \leqslant \rho,\ \forall t \in [0,T]$,

$$\begin{aligned} X_i^\infty(\omega,t) = X_i^\infty(\omega,0) + W_i(\omega,t) &- \frac{1}{2}\int_0^t \sum_{j\in\mathbb{N}} \nabla\varphi\big(X_i^\infty(\omega,s) - X_j^\infty(\omega,s)\big)\,ds \\ &+ \int_0^t \sum_{j\in\mathbb{N}} \big(X_i^\infty(\omega,s) - X_j^\infty(\omega,s)\big)\,dL_{ij}^\infty(\omega,s). \end{aligned}$$

Step 2. Let

$$\Omega(T) = \limsup_{\rho\to+\infty} \Omega\left(\rho, \left[\rho + \frac{\varepsilon_0}{4}\rho T + R\right]\right) \quad \text{and} \quad \Omega_1 = \bigcap_{T\in\mathbb{N}^*} \Omega(T) \subset \Omega_0.$$

$$\begin{aligned} P\big(\Omega(T)^c\big) &\leqslant \sum_{\rho_0\geqslant 1} \inf_{\rho\geqslant\rho_0} P\left(\Omega\left(\rho, \left[\rho + \frac{\varepsilon_0}{4}\rho T + R\right]\right)^c\right) \\ &\leqslant \sum_{\rho_0\geqslant 1} \inf_{\rho\geqslant\rho_0} C_5\left[\rho + \frac{\varepsilon_0}{4}\rho T + R\right]^d \mathrm{e}^{-C_6\rho} = 0 \end{aligned}$$

thanks to Lemma 7.2. Since this holds for each $T \in \mathbb{N}^*$, one has $P(\Omega_1) = 1$.

By definition of Ω_1: $\forall \omega \in \Omega_1,\ \forall T \in \mathbb{N}^*,\ \forall \rho_0 \in \mathbb{N}^*,\ \exists \rho \geqslant \rho_0,\ \forall i$ such that $|X_i^\infty(\omega,0)| \leqslant \rho,\ \forall t \in [0,T]$

$$\begin{aligned} X_i^\infty(\omega,t) = X_i^\infty(\omega,0) + W_i(\omega,t) &- \frac{1}{2}\int_0^t \sum_{j\in\mathbb{N}} \nabla\varphi\big(X_i^\infty(\omega,s) - X_j^\infty(\omega,s)\big)\,ds \\ &+ \int_0^t \sum_{j\in\mathbb{N}} \big(X_i^\infty(\omega,s) - X_j^\infty(\omega,s)\big)\,dL_{ij}^\infty(\omega,s). \end{aligned}$$

Since $P(\Omega_1)=1$, this proves that X^∞ is a solution of $(\mathcal{E})$.

Step 3.

To obtain a solution of $(\mathcal{E})$ with deterministic initial configuration, we disintegrate the solution X^∞ obtained above. This provides with a solution of $(\mathcal{E})$ for each configuration in a set $\underline{\mathcal{A}}_\mu$ with $\mu(\underline{\mathcal{A}}_\mu)=1$, and then globally for each configuration in

$$\underline{\mathcal{A}}=\bigcup_{z<z_c}\ \bigcup_{\mu\in\mathcal{G}(z)}\underline{\mathcal{A}}_\mu. \qquad \square \tag{20}$$

PROPOSITION 7.4. – *The process* $(X_i^\infty(t), L_{ij}^\infty(t),\ i,j\in\mathbb{N},\ t\in\mathbb{R}^+)$ *is the unique Markovian solution of equation* $(\mathcal{E})$ *inside of the class of paths* $\mathcal{C}$ *defined as follows*:

$X\in\mathcal{C}(\mathbb{R}^+,\mathcal{A})$ *belongs to* $\mathcal{C}$ *if there exists* $\varepsilon>0$ *and* $p\in\mathbb{N}^*$ *such that for all* $T,\rho,m_0\in\mathbb{N}^*$ *there exists an integer* $m\geqslant m_0$, *a sequence* $0=t_0<t_1<\cdots<t_{m'}=T$ *in* $\mathbb{Q}$ *verifying* $t_{k+1}-t_k\leqslant 1/m$ *and bounded open sets* $C_0,C_1,\ldots,C_{m'-1}$ *in* $\mathbb{R}^d$ *which satisfy*

$$\begin{aligned}B(0,\rho+mT)\subset C_{m'-1}\subset B(C_{m'-1},\varepsilon)\subset C_{m'-2}\cdots\\ \cdots\subset C_0\subset B\big(0,\rho+mT+m^p\big)\cdots\end{aligned}$$

and $\forall k\in\{0,\ldots,m'-1\}$,

$$d\big(\{x_j(u),\ j\in\mathbb{N}^*, u\in[t_k,t_{k+1}]\},\partial C_k\big)\geqslant\frac{R}{2}+\frac{\varepsilon}{4}.$$

Proof of Proposition 7.4. – We first check that for $\omega\in\Omega_1,\ X^\infty(\omega,\cdot)\in\mathcal{C}$.

We choose $\varepsilon=\varepsilon_0$ and $p=2$. For each T,ρ and m_0 in $\mathbb{N}^*$, one may find $l\geqslant l_0(\omega,T,\rho)$ large enough to have $m(\rho,l,T)\geqslant m_0$ and $m(\rho,l,T)\geqslant T(R+\varepsilon_0)/2$. Then $m=m(\rho,l,T)$, $m'=mT$, $t_k=k/m$ and

$$\begin{aligned}C_k=B&\left(0,\rho+mT+m^2-(k+1)\frac{m}{T}+\frac{R+\varepsilon_0}{2}\right)\\ &\cup\bigcup_{i\in J_k^l}B\left(X_i^\infty\left(\omega,\frac{k}{m}\right),\frac{R+\varepsilon_0}{2}\right)\end{aligned}$$

are convenient choices.

The basic idea of the proof of uniqueness inside of the class $\mathcal{C}$ is to decompose the time in a union of intervals $[k/m, (k+1)/m]$, on which each coordinate of the process is the unique solution of a finite-dimensional stochastic differential equation like (15). It is then a direct generalization of Lemma 5.4 in [8]. So we omit it.

Markovianity: It is clear that if a process Y belongs to $\mathcal{C}$, all the time-translated processes $\theta_s Y = Y(s+.)$, $s > 0$, belong to $\mathcal{C}$ too. Moreover, let $X^\infty(t,x)$ denotes the solution of $(\mathcal{E})$ for an initial condition $x \in \mathcal{A}$. Then, since $\theta_s X^\infty(.,x)$ belongs to $\mathcal{C}$ and is a solution of equation $(\mathcal{E})$, by the uniqueness property proved above, the following distributions coincide: the conditional law of $X^\infty(.+s,x)$ given the σ-algebra $\mathcal{F}_s$ and the law of $X^\infty(., X^\infty(s,x))$. This is the Markov property. □

PROPOSITION 7.5. – *If μ, the law of $X^\infty(0)$, belongs to $\mathcal{G}(z)$, then the process X^∞ solution of equation $(\mathcal{E})$ is a reversible process.*

Proof of Proposition 7.5. – Since X^∞ is Markovian, it is enough to prove that, for f, g bounded continuous functions on $\mathcal{M}$ with compact support and $s, t > 0$,

$$E\big(f\big(X^\infty(s)\big)g\big(X^\infty(t)\big)\big) = E\big(f\big(X^\infty(t)\big)g\big(X^\infty(s)\big)\big). \tag{21}$$

Equality (21) holds if the following equality holds:

$$\lim_{l\to+\infty} E\big(f\big(X^l(s)\big)g\big(X^l(t)\big) - f\big(X^l(t)\big)g\big(X^l(s)\big)\big) = 0.$$

Like in the proof of (11), we go back to the process $X^{l,\eta}$, which, by Proposition 4.1, is reversible when its initial distribution is $\mu^{l,\eta}$:

$$\begin{aligned}
&\big|E\big(f\big(X^l(s)\big)g\big(X^l(t)\big) - f\big(X^l(t)\big)g\big(X^l(s)\big)\big)\big| \\
&\quad = \bigg|\iint E\big(f\big(X^l(s)\big)g\big(X^l(t)\big) - f\big(X^l(t)\big)g\big(X^l(s)\big) \mid X^l(0) = \\
&\quad \xi_\Lambda \eta_{\Lambda^c}\big)\, d\mu(\xi|\eta_{\Lambda^c}) d\mu(\eta)\bigg| \\
&\quad = \bigg|\iint E\big(f\big(X^{l,\eta}(s)\big)g\big(X^{l,\eta}(t)\big) - f\big(X^{l,\eta}(t)\big)g\big(X^{l,\eta}(s)\big) \mid X^{l,\eta}(0) = \\
&\quad \xi_\Lambda\big)\, d\mu(\xi|\eta_{\Lambda^c}) d\mu(\eta)\bigg|
\end{aligned}$$

$$\leqslant \left| \int_{\mathcal{A}} \int_{\mathcal{A}} f\big(X(s)\big) g\big(X(t)\big) - f\big(X(t)\big) g\big(X(s)\big)\, dQ^{l,\eta}(X)\, d\mu(\eta) \right|$$

$$+ 2 \sup_{\xi \in \mathcal{A}} \big| f(\xi) \big| \sup_{\xi \in \mathcal{A}} \big| g(\xi) \big| \int_{\mathcal{A}} \left(1 - \frac{Z^{\Lambda,\eta}}{Z^{l,\eta}} \right) d\mu(\eta).$$

The first term of the sum vanishes and the second term tends to zero. □

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